MATH410 Review Sheet
(Includes “Theorems to Know”)

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Contents

1 Definitions

1.1 Foundations

Super Important Definition (Supremum and Infimum)

Every set S that is bounded from above has a *least upper bound*, called the *supremum* of S, denoted $\sup S$, such that it is the smallest number in $\mathbb{R}$ such that no element of S is greater than it.

Every set S that is bounded from below has a *greatest lower bound*, called the *infimum* of S, denoted $\inf S$, such that it is the largest number in $\mathbb{R}$ such that no element of S is smaller than it.

Sets that are bounded (i.e. bounded from below and above) must have both.

> An important property of the supremum is that
>
> **Theorem (Element Between Supremum)**
>
> Let the set $S \subseteq \mathbb{R}$ be nonempty and bounded. Then $\forall \epsilon > 0$, $\exists x \in S$ such that
>
> $$(\sup S) - \epsilon < x \leq \sup S.$$

Definition (Denseness)

A set S is said to be *dense* in $\mathbb{R}$ if it is the case that, for any $x_1, x_2 \in \mathbb{R}$ for which $x_1 < x_2$, $\exists s \in S$ s.t.

$$x_1 < s < x_2.$$

In other words, a set S is dense in $\mathbb{R}$ if it is the case that every open interval (a, b) contains a point in S.

> $\mathbb{Q}$ is dense, and so is $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ (the set of irrationals).

Super Important Definition (Triangle Inequality)

For any pair of numbers a and b in $\mathbb{R}$,

$$|a + b| \leq |a| + |b|$$

The **Reverse Triangle Inequality** states

$$|a - b| \geq ||a| - |b||$$

1.2 Sequences

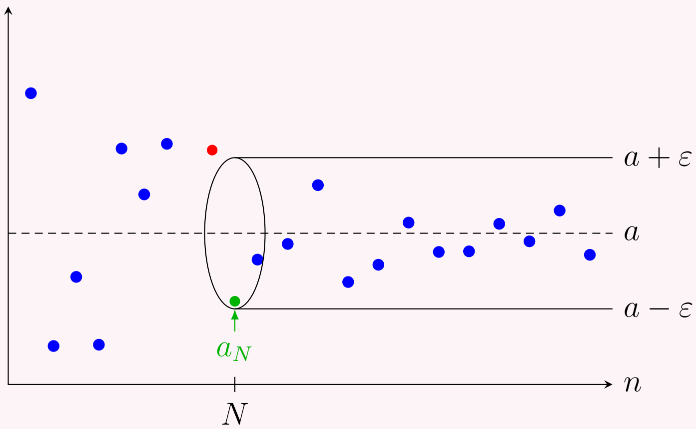
Super Important Definition (Convergence)

A sequence is said to *converge* to the number a provided that for every positive number ϵ there is an index N such that

$$|a_n - a| < \epsilon \quad \text{for all indices } n \geq N.$$

Thus a sequence $\{a_n\}$ is defined to converge to the number a provided that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$a - \epsilon < a_n < a + \epsilon \quad \text{for all indices } n \geq N.$$



Algebraic Limit Properties

- $\lim_{n\to\infty} [a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\forall \alpha \in \mathbb{R}\ \{a_n\} \to a \implies \{\alpha a_n\} \to \alpha a$
- $\lim_{n\to\infty} [a_n b_n] = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
- $\forall n \in \mathbb{N}\ b_n \neq 0$ and $b \neq 0, \{a_n\} \to a \implies \lim_{n\to\infty} \left[\frac{a_n}{b_n}\right] = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

Definition (Comparison Lemma)

Let the sequence $\{a_n\}$ converge to the number a. Then the sequence $\{b_n\}$ converges to the number b if $\exists C \geq 0$ and $\exists N_1 \in \mathbb{N}$ such that

$$|b_n - b| \leq C|a_n - a| \quad \text{for all indices } n \geq N_1.$$

Definition (Boundedness of Sets)

A set S is said to be bounded provided it is bounded above and below, that is,

$$\exists \sup S \quad \text{and} \quad \exists \inf S$$

which also implies that $\exists M \geq 0$ such that

$$|x| \leq M \quad \text{for all points } x \text{ in } S.$$

> You should think of the 2nd assertion as "the sandwich inequality". The assertion "$\forall x \in S, \exists M \geq 0$ s.t. $|x| \leq M$" just means that one single magnitude M can be used to draw vertical lines on the number line at $+M$ and $-M$ where every element of S lies between. It does not mean that M has to be the supremum or the infimum; in fact, this could be impossible if, say, $\sup S = 3$ and $\inf S = -7$. All we can say is that there is an $M = 7$ such that the lines $x = 7$ and $x = -7$ sandwich the entirety of S on the number line.

Definition (Boundedness of Convergent Sequences)

A sequence $\{a_n\}$ is said to be *bounded* if there is a number $M \in \mathbb{R}$ such that

$$|a_n| \leq M \quad \text{for every index } n.$$

> Every convergent sequence is bounded.

Definition (Open)

The set $S \subseteq \mathbb{R}$ is open if $\forall x \in S, \exists \epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subseteq S.$$

Definition (Closed Sets)

A subset S of $\mathbb{R}$ is said to be *closed* provided that if $\{a_n\}$ is a sequence in S that converges to a number a, then the limit a also belongs to S.
Alternatively, a subset S is closed if and only if its complement $\mathbb{R} \setminus S$ is open.

Definition (Subsequences)

Consider a sequence $\{a_n\}$. Let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing; that is,

$$n_1 < n_2 < n_3 < \cdots .$$

Then the sequence $\{b_k\}$ defined by

$$b_k = a_{n_k} \quad \text{for every index } k$$

is called a *subsequence* of the sequence $\{a_n\}$.

> A subsequence is just: take some numbers which are strictly increasing, e.g. 3, 7, 15, 27, 105, ... and take only the elements of $\{a_n\}$ with $n = 3, 7, 15, 27, 105, ...$ infinitely.
>
> - Every subsequence of $\{a_n\} \to a$ also converges to a.
> - Every sequence has a monotone subsequence.
> - Every bounded sequence has a convergent subsequence.

Definition (Sequential Compactness)

A set of real numbers S is said to be *sequentially compact* provided that every sequence $\{a_n\}$ in S has a subsequence that converges to a point that belongs to S.

> If S is a subset of $\mathbb{R}$ that is closed and bounded, S is sequentially compact.

Definition (Open Covers)

Let $S \subset R$. An open cover of S is a collection $\mathcal{O}$ of open sets such that

$$S \subset \bigcup_{U \in \mathcal{O}} U.$$

In plain English, $\mathcal{O}$ is an open cover of S if every point of S is contained in at least one of the open sets in $\mathcal{O}$.

1.3 Continuity

Definition (Continuity - Pointwise)

A function $f : D \to \mathbb{R}$ is said to be *continuous at the point* x_0 in D if it is the case that whenever $\{x_n\}$ is a sequence in D that converges to x_0, the image sequence $\{f(x_n)\}$ converges to $f(x_0)$. The function $f : D \to \mathbb{R}$ is said to be *continuous* provided that it is continuous at every point in D.
In other words,
A function is continuous at $x_0 \in D$ if

$$\{x_n\} \to x_0 \implies \{f(x_n)\} \to f(x_0)$$

Definition (Continuity - The ϵ-δ Criterion)

A function $f : D \to \mathbb{R}$ is said to be *continuous* at $x_0 \in D$ if, $\forall \epsilon > 0$, $\exists \delta > 0$ such that for all $x \in D$,

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

Definition (Uniform Continuity)

A function $f : D \to \mathbb{R}$ is said to be uniformly continuous provided that whenever $\{u_n\}$ and $\{v_n\}$ are sequences in D such that

$$\lim_{n\to\infty} [u_n - v_n] = 0,$$

then

$$\lim_{n\to\infty} [f(u_n) - f(v_n) = 0].$$

> $\{u_n\}$ and $\{v_n\}$ don't have to converge in D; the concept of uniform continuity of f on D means "the difference $f(u) - f(v)$ becomes arbitrarily small for any two points u and v in D that are sufficiency close to each other, no matter where the two points are located in the domain."

Definition (Uniform Continuity - ϵ-δ)

A function $f : D \to \mathbb{R}$ is said to be uniformly continuous when, $\forall \epsilon > 0$, $\exists \delta > 0$ such that for all $c, d \in D$,

$$|c - d| < \delta \implies |f(c) - f(d)| < \epsilon.$$

> This is the same thing as satisfying the ϵ-δ criteria for pointwise continuity, except on the entire domain D.

Definition (Monotone Functions)

The function $f : D \to \mathbb{R}$ is called *monotonically increasing* if for all $c, d \in D$, we have

$$c < d \implies f(c) \leq f(d)$$

The function $f : D \to \mathbb{R}$ is called *monotonically decreasing* if for all $c, d \in D$, we have

$$c < d \implies f(c) \geq f(d)$$

Either of these functions are called *monotone*. We then say monotonicity is *strict* if the $\geq$ and $\leq$ relations are modified to $>$ and $<$ relations.

Monotone Continuity
A function $f : D \to \mathbb{R}$ that is monotone is continuous if its image $f(D)$ is an interval.

Monotone Injectivity
Strictly monotone functions are injective (one-to-one) and thus have inverse functions.

Monotone Inverse Continuity
Let I be an interval and suppose the function $f : I \to \mathbb{R}$ is strictly monotone. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.

1.4 Derivatives

Super Important Definition (Derivatives)

For a number x_0, an open interval $I = (a, b)$ that contains x_0 is called a neighborhood of x_0. Let I be a neighborhood of x_0. Then the function $f : I \to \mathbb{R}$ is said to be differentiable at x_0 provided that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, in which case we denote this limit by $f'(x_0)$ and call it the derivative of f at x_0; that is,

$$f'(x_0) \equiv \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If the function $f : I \to \mathbb{R}$ is differentiable at every point in I, we say that f is *differentiable* and call the function $f' : I \to \mathbb{R}$ the *derivative* of f.

Corollaries

- $f(x) = x^n \implies f'(x) = nx^{n-1}$
- $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$
- Differentiable functions are continuous at the x_0 they are differentiable at.
- Let I be a neighborhood of x_0 and let the function $f : I \to \mathbb{R}$ be strictly monotone and continuous. Suppose that f is differentiable at x_0 and that $f'(x_0) \neq 0$. Define $J = f(I)$ (the image of f). Then the inverse $f^{-1} : J \to \mathbb{R}$ is differentiable at the point $y_0 = f(x_0)$ and
$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$
- Let I be an open interval and suppose that the function $f : I \to R$ is strictly monotone and differentiable with a nonzero derivative at each point in I. Define $J = f(I)$. Then the inverse function $f^{-1} : J \to \mathbb{R}$ is differentiable and
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \quad \text{for all } x \text{ in } J.$$
- I an open interval, $f : I \to \mathbb{R}$ differentiable, $f'(x) > 0 \quad \forall x \in I \implies f$ is strictly increasing.

1.5 Integration

Definition (Darboux Sums)

Let I be an interval, and $f : I \to \mathbb{R}$ be a bounded function. Let $P = \{x_0, ..., x_n\}$ be a partition of the interval I. Then, the *upper and lower Darboux sums* of f over P are the sums

$$U(f, P) = \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1})$$

$$L(f, P) = \sum_{i=1}^{n} \left(\inf_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1})$$

> For every partition P of I,
>
> $$L(f, P) \leq \int_a^b f \leq U(f, P).$$

Definition (Upper and Lower Integrals)

Suppose the function $f : [a, b] \to \mathbb{R}$ is bounded. Let $\mathcal{P}$ be the set of all partitions of the interval $[a, b]$. Then we define the lower integral of f on $[a, b]$, which we denote by $\underline{\int}_a^b f$, by

$$\underline{\int}_a^b f \equiv \sup_{P \in \mathcal{P}} L(f, P).$$

We define the upper integral of f on $[a, b]$, which we denote by $\overline{\int}_a^b f$, by

$$\overline{\int}_a^b f \equiv \inf_{P \in \mathcal{P}} U(f, P).$$

Definition (Integrability)

Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded. Then we say that $f : [a, b] \to \mathbb{R}$ is *integrable*, or that f is integrable on $[a, b]$, if it is the case that

$$\underline{\int}_a^b f = \overline{\int}_a^b f$$

When this is so, the integral of the function $f : [a, b] \to \mathbb{R}$, denoted by $\int_a^b f$, is defined by

$$\int_a^b f \equiv \underline{\int}_a^b f = \overline{\int}_a^b f$$

Definition (Regular Partitions)

For a natural number n, the partition $P = \{x_0, ..., x_n\}$ of the interval $[a, b]$ defined by

$$x_i = a + i\frac{(b-a)}{n} \quad \text{for } 0 \leq i \leq n$$

is called the *regular partition* of $[a, b]$ into n partition intervals. All partition intervals have the same length, $(b-a)/n$.

Definition (Gap of a Partition)

For a partition $P = \{x_0, ..., x_n\}$ of the interval $[a, b]$, we define the *gap of* P, denoted by $\operatorname{gap} P$, to be the length of the largest partition interval of P; that is,

$$\operatorname{gap} P \equiv \max_{1 \leq i \leq n} [x_i - x_{i-1}].$$

> If, for $\epsilon > 0$, $\operatorname{gap} P < \epsilon$, then it means that every single partition interval of P is smaller in length than ϵ. (This means we can make partition intervals arbitrarily small, and use that to prove convergence of Darboux sums into integrals).

1.6 Taylor Polynomials

Definition (Contact Order)

Let I be a neighborhood of the point x_0. Two functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are said to *have contact order* 0 at x_0 if $f(x_0) = g(x_0)$. For $n \in \mathbb{N}$, the functions f and g are said to *have contact order* n at x_0 if $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ have n derivatives and

$$f^{(k)}(x_0) = g^{(k)}(x_0) \quad \text{for } 0 \leq k \leq n.$$

Definition (Taylor Polynomial)

Let I be a neighborhood of the point x_0 and let $n \in \mathbb{Z}_0^+$. Suppose that the function $f : I \to \mathbb{R}$ has n derivatives. Then there is a unique polynomial of degree at most n that has contact order n with the function $f : I \to \mathbb{R}$ at x_0. This polynomial is defined by the formula

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Such a polynomial is called the *nth Taylor polynomial* for f at x_0.

Definition (Taylor Series Expansion)

Let I be a neighborhood of the point x_0 and suppose that the function $f : I \to \mathbb{R}$ has derivatives of all orders. The nth Taylor polynomial for f is defined by

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

If x is a point in I at which

$$\lim_{n\to\infty} p_n(x) = f(x),$$

we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Such a formula is called a *Taylor series expansion* of f about x_0, and it holds at x if and only if

$$\lim_{n\to\infty} [f(x) - p_n(x)] = 0.$$

1.7 Series, Sequences of Functions

Definition (Cauchy Convergence Criterion - Sequences)

A sequence of numbers $\{a_n\}$ is said to be a *Cauchy sequence* if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \text{ and } m \geq N \implies |a_n - a_m| < \epsilon$$

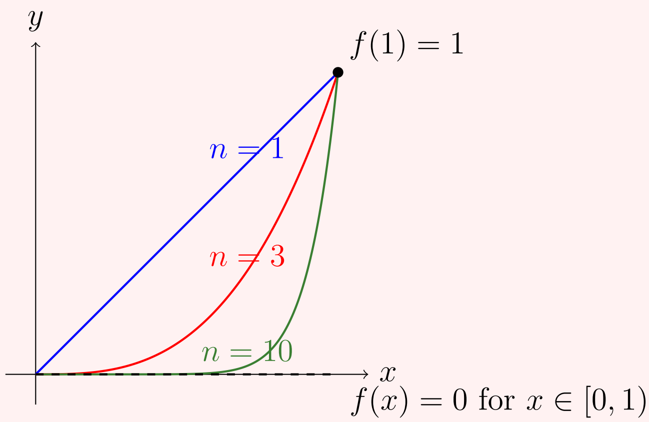
Properties of Cauchy Sequences

- Every convergent sequence is Cauchy.
- Every Cauchy sequence is bounded.
- **Theorem.** *A sequence of numbers converges if and only if it is a Cauchy sequence.*

Definition (Pointwise Convergence)

Given a function $f : D \to \mathbb{R}$ and a sequence of functions $\{f_n : D \to \mathbb{R}\}$, we say that the sequence $\{f_n : D \to \mathbb{R}\}$ *converges pointwise* to $f : D \to \mathbb{R}$, or that $\{f_n\}$ *converges pointwise* on D to f, if $\forall x \in D$,

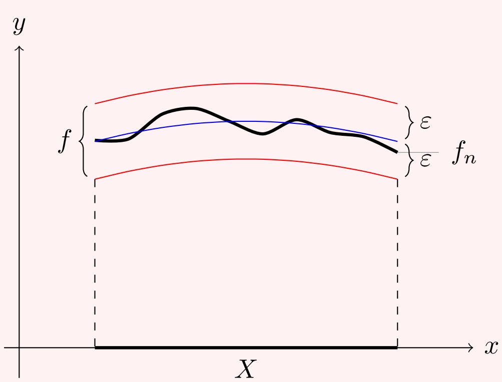
$$\lim_{n\to\infty} f_n(x) = f(x).$$



Definition (Uniform Convergence)

Given a function $f : D \to \mathbb{R}$ and a sequence of functions $\{f_n : D \to \mathbb{R}\}$, we say that the sequence $\{f_n : D \to \mathbb{R}\}$ *converges uniformly* to $f : D \to \mathbb{R}$, or that $\{f_n\}$ *converges uniformly* on D to f, if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\forall x \in D$,

$$|f(x) - f_n(x)| < \epsilon.$$



Definition (Uniformly Cauchy)

The sequence of functions $\{f_n : D \to \mathbb{R}\}$ is said to be *uniformly Cauchy*, or $\{f_n\}$ is said to be *uniformly Cauchy* on D, if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\forall x \in D$, $\forall k \in \mathbb{N}$,

$$|f_{n+k}(x) - f_n(x)| < \epsilon.$$

Theorem. The sequence of functions $\{f_n : D \to \mathbb{R}\}$ converges uniformly to a function $f : D \to \mathbb{R}$ if and only if the sequence $\{f_n : D \to \mathbb{R}\}$ is uniformly Cauchy.

Definition (Power Series)

Given a sequence of real numbers $\{c_k\}$ indexed by the nonnegative integers, we define the *domain of convergence* of the series $\sum_{k=0}^{\infty} c_k x^k$ to be the set of all numbers x such that the series $\sum_{k=0}^{\infty} c_k x^k$ converges. Denote the domain of convergence by D. We then define a function $f : D \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty}\left[\sum_{k=0}^{\infty} c_k x^k\right] = \sum_{k=0}^{\infty} c_k x^k$$

2 Theorems

2.1 "Theorems to Know"

Theorem 1.5 (The Archimedean Property)

The following two equivalent properties hold:

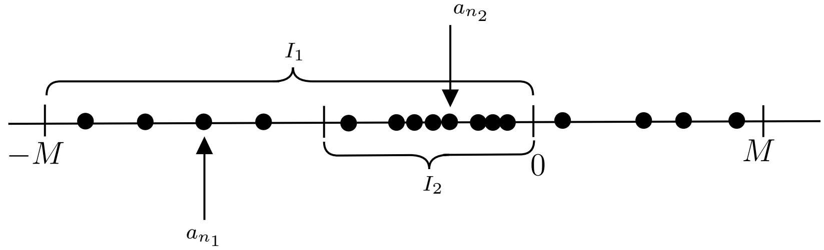
1. For any positive number c, there is a natural number n such that $n > c$.
2. For any positive number ϵ, there is a natural number n such that $\frac{1}{n} < \epsilon$.

> This basically just says that the natural numbers are unbounded - you can always find a natural number bigger than any number, and you can always find a number that is the reciprocal of a natural number that is smaller than any number.

Theorem 2.36 (Bolzano-Weierstrass Theorem/Sequential Compactness Theorem)

Let a and b be numbers such that $a < b$. Then the interval $[a, b]$ is sequentially compact; that is, every sequence in $[a, b]$ has a subsequence that converges to a point in $[a, b]$.

> Every bounded sequence has a convergent subsequence.
> $\Longrightarrow$ Every convergent sequence has a convergent subsequence, and all of those convergent subsequences have the same limit.

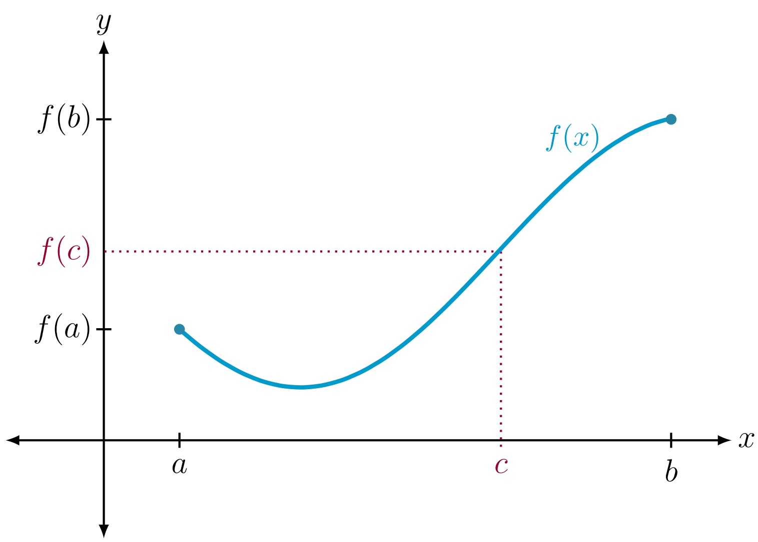


Theorem 3.11 (Intermediate Value Theorem)

Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Let c be a number strictly between $f(a)$ and $f(b)$; that is,

$$f(a) < c < f(b) \quad \text{or} \quad f(b) < c < f(a).$$

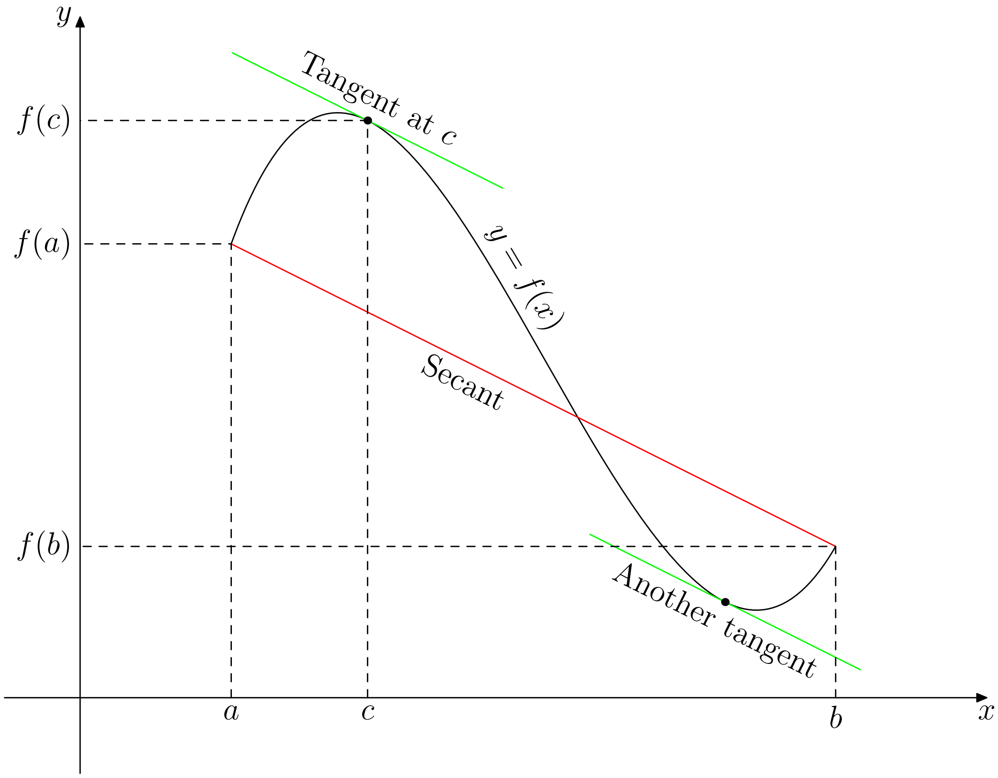
Then there is a point x_0 in the open interval (a, b) at which $f(x_0) = c$.



Theorem 4.18 (Mean Value Theorem)

Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that the restriction of f to the open interval (a, b) is differentiable. Then there is a point x_0 in the open interval (a, b) at which

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

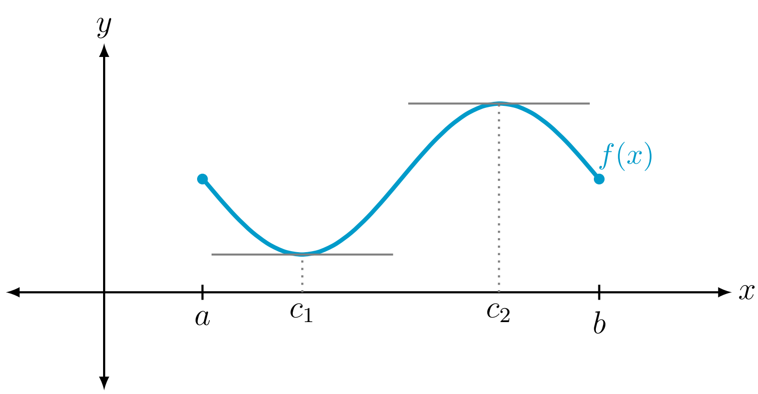


Rolle's Theorem
Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that the restriction of f to the open interval (a, b) is differentiable. Assume, moreover, that

$$f(a) = f(b).$$

Then there is a point x_0 in the open interval (a, b) at which

$$f'(x_0) = 0.$$



Theorem 6.8 (Archimedes-Riemann Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if there is a sequence of partitions $\{P_n\}$ of the interval $[a, b]$ such that

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0.$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f \quad \text{and} \quad \lim_{n\to\infty} U(f, P_n) = \int_a^b f.$$

Corollaries

- A monotonically increasing function $f : [a, b] \to \mathbb{R}$ is integrable.
- Step functions $f : [a, b] \to \mathbb{R}$ are integrable.
- Integrals are additive, monotone ($f(x) \leq g(x) \implies \int_a^b f \leq \int_a^b g$), and linear.
- Suppose the functions $f : [a, b] \to \mathbb{R}$ and $|f| : [a, b] \to \mathbb{R}$ are integrable. Then

$$\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx$$

Proof. $\forall x \in [a, b]$,

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

Thus, using monotonicity and linearity,

$$-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)| dx$$

which is equivalent. □

- A continuous function on a closed bounded interval $f : [a, b] \to \mathbb{R}$ is integrable.
- The value of the integral of a bounded on closed and continuous on open function exists and does not depend on the values of f at the endpoints on the integral.

Theorem 6.22 (FTC I: Integrating Derivatives)

Let the function $F : [a,b] \to \mathbb{R}$ be continuous on the closed interval $[a,b]$ and be differentiable on the open interval (a,b). Moreover, suppose that its derivative

$$F' : (a,b) \to \mathbb{R} \text{ is both continuous and bounded.}$$

Then

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a).$$

Theorem 6.29 (FTC II: Differentiating Integrals)

Suppose that the function $f : [a,b] \to \mathbb{R}$ is continuous. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^b f\right] = f(x) \quad \text{for all } x \text{ in } (a,b).$$

Theorem 7.13 (Riemann Sum Convergence Theorem)

Suppose that the function $f : [a,b] \to \mathbb{R}$ is integrable. For each natural number n, let P_n be a partition of $[a,b]$ and let $R(f, P_n, C_n)$ be a Riemann sum. If

$$\lim_{n\to\infty} \operatorname{gap} P_n = 0,$$

then

$$\lim_{n\to\infty} R(f, P_n, C_n) = \int_a^b f.$$

Theorem 8.8 (Lagrange Remainder Theorem)

Let I be a neighborhood of the point x_0 and let n be a nonnegative integer. Suppose that the function $f : I \to \mathbb{R}$ has $n+1$ derivatives. Then for each point $x \neq x_0$ in I, there is a point c strictly between x and x_0 such that

$$f(x) = p_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

> This means that
>
> $$p_n(x) = f(x) - R_n(x)$$
>
> meaning that for any natural number n, the n-th Taylor polynomial is at most $R_n(x)$ off from the true value of the function at any x.

Theorem 8.14 (Taylor Series Convergence Theorem)

Let I be a neighborhood of the point x_0 and suppose that the function $f : I \to \mathbb{R}$ has derivatives of all orders. Suppose also that there are positive numbers r and M such that the interval $[x_0 - r, x_0 + r]$ is contained in I and that for every natural number n and every point x in $[x_0 - r, x_0 + r]$,

$$|f^{(n)}(x)| \leq M^n.$$

Then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \quad \text{if } |x - x_0| \leq r.$$

Theorem 9.31 (Uniform Limit Continuity Theorem)

Suppose that $\{f_n : D \to \mathbb{R}\}$ is a sequence of continuous functions that converges uniformly to the function $f : D \to \mathbb{R}$. Then the limit function $f : D \to \mathbb{R}$ is also continuous.

> **Note:** this theorem is strictly one-way! It is not true that a continuous limit function implies a continuous sequence of functions which converges to it.
> A similar theorem exists for integrability - $\{f_n\}$ integrable $\implies$ f integrable.

Theorem 9.41 (Term-by-Term Differentiation of Power Series)

Let r be a positive number such that the interval $(-r, r)$ lies in the domain of convergence of the series $\sum_{k=0}^{\infty} c_k x^k$. Define

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \quad \text{if } |x| < r.$$

Then the function $f : (-r, r) \to \mathbb{R}$ has derivatives of all orders. For each natural number n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[f(x)] = \sum_{k=0}^{\infty} \frac{\mathrm{d}^n}{\mathrm{d}x^n}[c_k x^k] \quad \text{if } |x| < r,$$

so that, in particular, for each natural number n,

$$\frac{f^{(n)}(0)}{n!} = c_n.$$

> Basically, you can term-by-term differentiate a power series. You can swap the $\sum$ and the $\mathrm{d}/\mathrm{d}x$ if it's a power series.

3

Lemmas

Lemma (Factorial is Faster than Polynomial)

For any number c,

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0.$$

Proof. Choose k to be a naural number such that $k \geq 2|c|$. Then if $n \geq k$,

$$\begin{aligned} 0 &\leq \left|\frac{c^n}{n!}\right| \\ &= \left[\frac{|c|}{1}\cdots\frac{|c|}{k}\right]\left[\frac{|c|}{k+1}\cdots\frac{|c|}{n}\right] \\ &\leq |c|^k\left(\frac{1}{2}\right)^{n-k} \\ &= |c|^k 2^k\left(\frac{1}{2}\right)^n \end{aligned}$$

Since $\lim_{n\to\infty}(1/2)^n = 0$, the limit equals 0. □